# A Simplified Model of an Activated Sludge Process with a Plug-Flow Reactor

Jesús Zambrano*,[a], Bengt Carlsson[b], Stefan Diehl[c], Emma Nehrenheim[a]

[a]*School of Business, Society and Engineering*, Mälardalen University, Box 883, 72123 Västerås, Sweden.
[b]*Department of Information Technology*, Uppsala University, Box 337, 75105 Uppsala, Sweden.
[c]*Centre for Mathematical Sciences*, Lund University, Box 118, 22100 Lund, Sweden.
*Corresponding author e-mail: jesus.zambrano@mdh.se

*Abstract* — **The analysis of a simplified activated sludge process (ASP) with one main dissolved substrate and one main particulate biomass component has been conducted with respect to its steady-state. The ASP is formed by a plug-flow reactor (PFR) and a settler with the recycling going to the reactor. The biomass growth rate is described by a Monod function. For this process, it is not possible to get an explicit expression for the effluent substrate concentration when the process is subject to a fixed sludge age. However, in the normal case when the influent substrate concentration is much greater than the effluent substrate concentration, then an explicit approximation for the effluent as a function of the influent and the process parameters is obtained. This work includes numerical examples considering two models for the settler. One model is the ideal settler, which assumes a complete thickening of the activated sludge through the underflow of the settler. The other model takes into account hindered settling and sludge compression. Numerical results show the effectiveness and the limitations of the proposed solution under these scenarios.**

*Keywords* — **bioreactor, clarifier, sludge blanket, sludge age.**

## I. INTRODUCTION

Steady-state modeling and analysis of ASPs have been extensively studied during the last 50 years. One important reason is that the steady-state analysis of a dynamic model can provide initial values for process operation and optimization.

Generally, the mixing regime in an ASP reactor neither behaves as a completely stirred tank reactor (CSTR) nor as a plug-flow reactor (PFR), but in some sense in between [1]. In a CSTR, the reactor content is well stirred, so it is assumed that the concentration in the effluent is the same as in the reactor. In a PFR, the key assumption is that the fluid is perfectly mixed in the radial direction and in the axial direction only the transportation of the fluid is considered. Therefore, a PFR can be seen as a series of infinitely thin CSTRs, each with a uniform and different composition than the neighbouring one [2]. It is expected that a PFR with a volume smaller than several CSTRs in series will give the same performance [3].

Compared to the classical ASP configuration, i.e. ASP with one CSTR, an explicit (steady-state) solution for an ASP formed by a PFR seems to not be possible to obtain [4]. However, some attempts have been made in the analysis of this process. For example, some implicit and approximate expressions for the effluent substrate were presented in [5], where the expressions were compared with numerical solutions. Computer techniques to solve the problem of a PFR in an ASP when considering the PFR as a large number of bioreactors in series are shown in [6]. Design graphs and numerical examples were presented as guidelines to size the process. On the other hand, a study of an ASP formed by a PFR could be seen as an approximation of an ASP with several CSTRs in series [7], [8].

A study of the relationship between the influent and effluent of an ASP formed by one and two CSTRs in series was presented in [8]. The study mentions that it does not seem possible to find explicit solutions for the effluent substrate concentration for two or more bioreactors. That work was the motivation that led to the development of [4] and the current study.

A steady-state analysis of an ASP formed by using a PFR and a settler was recently studied in [4]. The study considers and compares two different settler models. One is the ideal settler, which assumes an unlimited flux capacity, i.e. the settler is always considered to be overdimensioned. The other model, recently published in [9], here referred to as DZC settler model, includes hindered and compressive settling, which means that a limited flux capacity is modelled. Both numerical and, in some cases, analytical results are obtained. A comparison with an ASP formed by a single CSTR is also shown in [4].

In the present work, we continue the analysis of an ASP consisting of a PFR and a settler. For the ideal settler case, the steady-state solution is presented with explicit approximate formulas when the influent substrate concentration is much greater than the effluent substrate concentration. Under the same assumption, we also present an explicit formula for the effluent substrate concentration as a function of the influent substrate concentration when the sludge age is fixed. This formula can be used for both settler models.

The paper is organized as follows. A description of the ASP with a PFR and a settler is presented in Section II, including the steady-state mass balances and the definitions for the ideal and DZC settler models. In Sections III and IV, we review from [4] the equations describing the steady-state conditions of the ASP for both settler models. Section V contains an approximate explicit expression for the effluent substrate concentration. Numerical examples are shown in Section VI and conclusions are drawn in Section VII.
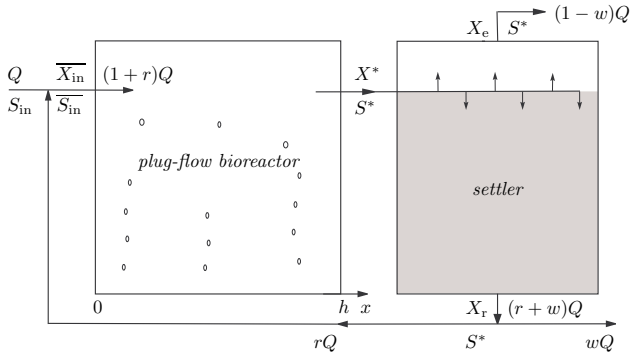
Fig. 1: The activated sludge process consisting of a PFR and a settler. The steady-state variables are shown as well as the horizontal $x$-axis of the PFR.

## II. THE ACTIVATED SLUDGE PROCESS

For the ASP we consider using a PFR coupled with a settler, see Figure 1, where the recycling flows to the reactor. The PFR has a constant vertical cross-sectional area $A$ and length $h$, so the volume is $V_R = Ah$. The variable $x$ is used to denote the horizontal axis in the PFR from the inlet ($x = 0$) to the outlet ($x = h$). Where the concentrations at location $x$ can be denoted as $S(x)$ and $X(x)$ in the PFR.

We assume two constituents, namely one particulate biomass $X$ and one dissolved substrate $S$. The influent volumetric flow rate and substrate concentration are denoted by $Q$ and $S_{in}$, respectively. It is assumed that no biomass is present in the influent ($X_{in} = 0$). The PFR input concentrations are denoted by $\overline{S_{in}}$ and $\overline{X_{in}}$, and the PFR outputs by $S^*$ and $X^*$. It is assumed that no reactions are taking place in the settler, so that only particulate biomass is influenced. The substrate concentration is thus unchanged and therefore equal to $S^*$ throughout the settler. The effluent at the top of the settler is $X_e$ and the recycle concentration is $X_r$. The recycle flow rate is $rQ$ and the waste flow rate is $wQ$, where $r > 0$ and $0 < w \leq 1$. The kinetics in the PFR are described by using the Monod function [10]

$$\mu(S) = \mu_{max} \frac{S}{K_s + S}, \tag{1}$$

where $\mu_{max}$ is the maximum specific growth rate and $K_s$ is the half-saturation constant. It is assumed that the biomass death is negligible.

The sludge age $\theta$ of the process is defined as the amount of biomass in the bioreactor divided by the removed biomass per unit time, and is expressed as

$$\theta = \frac{A \int_0^h X(x)dx}{wQX_r}. \tag{2}$$

### A. Mass balances and expression for the sludge age

The three mass balances of the process in steady state with $X_e = 0$ are

$$Q(1+r)\overline{S_{in}} = QS_{in} + rQS^*, \tag{3}$$
$$Q(1+r)\overline{X_{in}} = rQX_r, \tag{4}$$
$$Q(1+r)X^* = (r+w)QX_r. \tag{5}$$

Applying the conservation of mass in the PFR we get

$$\frac{Q(1+r)}{A}\frac{dS}{dx} = -\mu[S(x)]\frac{X(x)}{Y}, \tag{6}$$
$$\frac{Q(1+r)}{A}\frac{dX}{dx} = \mu[S(x)]X(x), \tag{7}$$

where $Y$ refers to the yield constant. The following boundary conditions hold: $X(0) = \overline{X_{in}}$, $S(0) = \overline{S_{in}}$, $X(h) = X^*$ and $S(h) = S^*$. By combining Equations (6) and (7) together with the boundary conditions, gives

$$\frac{Q(1+r)}{A}\frac{d(YS+X)}{dx} = 0$$
$$\implies Y\overline{S_{in}} + \overline{X_{in}} = YS(x) + X(x) = YS^* + X^*. \tag{8}$$

By solving for $X(x)$ in Equation (8) and substituting it into (6), using $V_R = Ah$ and integrating, we get the following equation for the PFR:

$$-Q(1+r)Y \int_{\overline{S_{in}}}^{S^*} \frac{d\sigma}{\mu(\sigma)\left[\overline{X_{in}} + Y(\overline{S_{in}} - \sigma)\right]} = V_R$$
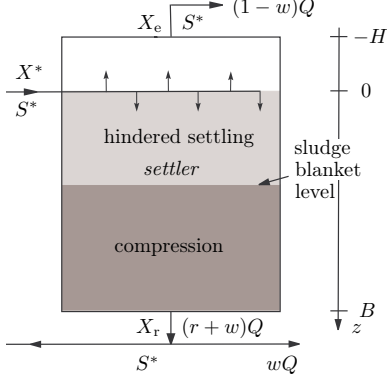$$\iff f(S^*, r, w) = V_R, \tag{9}$$

Fig. 2: The DZC settler. The steady-state variables are shown as well as the vertical $z$-axis of the settler.

where

$$f(S^*, r, w) = \frac{Q(1+r)}{\mu_{\max}} \left[ P \ln \left( \frac{a(S_{\mathrm{in}} + rS^*)}{S^*(1+r)} \right) + \ln(a) \right], \tag{10}$$

$$P = P(S^*, r, w) = \frac{K_s w(1+r)}{S_{\mathrm{in}}(r+w) - S^* r(1-w)},$$

$$a = a(r, w) = \frac{r+w}{r}.$$

We can obtain the sludge age by rewriting the integral in Equation (2) using Equation (6). Then we have (see [4])

$$\theta = \frac{A}{wQX_r} \left[ \frac{-Q(1+r)Y}{A} \int_{\overline{S}_{\mathrm{in}}}^{S^*} \frac{(K_s + \sigma)\mathrm{d}\sigma}{\mu_{\max}\sigma} \right] = \frac{1}{\mu_{\max}} \left[ 1 + \frac{(1+r)K_s}{(S_{\mathrm{in}} - S^*)} \ln \left( \frac{S_{\mathrm{in}} + rS^*}{S^*(1+r)} \right) \right]. \tag{11}$$

### B. The settler

**Ideal settler model**. For an ideal settler we assume that all the sludge fed to the settler will always pass through the thickening zone, regardless of the amount of incoming sludge and the recycle and waste flows. Although in many cases unrealistic, this model could work well when the settler is over-sized.

**DZC settler model**. The processes in the settler are described by a steady-state approximation of a partial differential equation (PDE) which includes a hindered settling velocity function and a compression function [11]. The behavior of a real settler can be divided into three qualitatively different operations: underloaded, overloaded and normal operation. By normal operation we mean that all the biomass fed to the settler is conveyed through the thickening zone and that there exists a sludge blanket in the thickening zone, see Figure 2. In this work we only study the steady-state solutions under normal operation and therefore set $X_e = 0$.

The following simple relationship is a reasonable approximation obtained from the steady-state solutions that have a sludge blanket in the thickening zone [9]

$$X_r = U_{z_{\mathrm{sb}}}(q) := X_{z_{\mathrm{sb}}}^{\infty} \left( 1 + \frac{\hat{q}_{z_{\mathrm{sb}}}}{q + \check{q}_{z_{\mathrm{sb}}}} \right), \tag{12}$$

where $q$ is the bulk velocity in the thickening zone, defined as

$$q = q(r, w, Q, A_S) := \frac{Q(r+w)}{A_S}, \tag{13}$$

where $X_{z_{\mathrm{sb}}}^{\infty}, \hat{q}_{z_{\mathrm{sb}}}$ and $\check{q}_{z_{\mathrm{sb}}}$ are parameters which depend on the chosen sludge blanket level $z_{\mathrm{sb}}$. $A_S$ is the settler constant horizontal cross-sectional area, see model details in [9].

### III. ASP WITH IDEAL SETTLER MODEL

#### A. Steady-state solutions

From the mass balances of the process (5), (8) and (9), the steady-state equations for an ASP with ideal settler can be expressed as (ignoring the variables $\overline{S_{\mathrm{in}}}$ and $\overline{X_{\mathrm{in}}}$; these can be obtained from (3) and (4))

$$S^* = S_{\mathrm{in}} - \frac{w}{Y} X_r, \tag{14}$$

$$X^* = \frac{r+w}{1+r} X_r, \tag{15}$$

$$V_R = f(S^*, r, w), \tag{16}$$

where $f(S^*, r, w)$ is given in Equation (10). To get the solution of these equations, Equation (16) is solved for $S^* = S^*(r, w)$, then Equation (14) gives $X_r = X_r(r, w)$ and Equation (15) gives $X^* = X^*(r, w)$. Note that all these variables are two-parameter solutions of the control variables $r, w$. Note also from Equations (9) and (10) that $S^*$ is expressed implicitly. If $S_{\mathrm{in}} \gg S^*$ is assumed, we have the following results.

**Theorem 1.** *Given an ASP with an ideal settler described by Equations* (14)–(16). *If* $S_{\mathrm{in}} \gg S^*$ *then the solution of Equations* (14)–(16) *can be expressed explicitly as*

$$S^* = S^*(r, w) = \frac{(r+w)S_{\mathrm{in}}}{r[(1+r)\exp(\beta) - (r+w)]}, \tag{17}$$

$$X_r = X_r(r, w) = \frac{Y}{w} (S_{\mathrm{in}} - S^*(r, w)), \tag{18}$$

$$X^* = X^*(r, w) = \frac{(r+w)Y}{(1+r)w} (S_{\mathrm{in}} - S^*(r, w)). \tag{19}$$

*where*

$$\beta = \frac{S_{\mathrm{in}}(r+w)}{K_s w(1+r)} \left[ \frac{V_R \mu_{\max}}{Q(1+r)} - \ln \left( \frac{r+w}{r} \right) \right],$$

*and when the denominator in Equation* (17) *is positive.*

*Proof.* The assumption implies that Equation (16) can be expressed as (cf. Equations (9) and (10))

$$\frac{Q(1+r)}{\mu_{\max}} \left[ \frac{1}{G} \ln \left( \frac{a(S_{\mathrm{in}} + rS^*)}{S^*(1+r)} \right) + \ln(a) \right] = V_R, \tag{20}$$

where

$$G = \frac{S_{\mathrm{in}}(r+w)}{K_s w(1+r)} \quad \text{and} \quad a = \frac{r+w}{r},$$

solving (20) for $S^*$ gives

$$G \left( \frac{V_R \mu_{\max}}{Q(1+r)} - \ln(a) \right) = \ln \left( \frac{a(S_{\mathrm{in}} + rS^*)}{S^*(1+r)} \right),$$

for simplicity we set $\beta = G\left(\frac{V_R\mu_{\max}}{Q(1+r)} - \ln(a)\right)$, then we have

$$\exp(\beta) = \frac{a(S_{in} + rS^*)}{S^*(1+r)} \Longleftrightarrow$$

$$S^*(1+r)\exp(\beta) = a(S_{in} + rS^*) \Longleftrightarrow$$

$$S^* = \frac{aS_{in}}{(1+r)\exp(\beta) - ar} \Longleftrightarrow$$

$$S^* = \frac{(r+w)S_{in}}{r\left[(1+r)\exp(\beta) - (r+w)\right]},$$

if the denominator is positive.

Once $S^*$ is obtained, $X_r$ and $X^*$ are given from Equations (14) and (15), respectively. $\qquad\square$

### B. Substrate input-output relationship for constant sludge age

The two-parameter solution of Equations (14)–(16) (or (17)–(19) in Theorem 1) means that two additional equations can be imposed to define the operation conditions. We are interested in investigating the steady-state solutions of the process for different values of $S_{in}$ for a constant sludge age $\theta_0$. For $S_{in}$ as a variable, we have six variables to take into consideration: $S^*, X^*, X_r, r, w$, and $S_{in}$. However, to get a one-parameter solution with $S_{in}$ as a parameter, we can add the following to Equations (14)–(16):

$$r = r_0, \tag{21}$$

$$\frac{1}{\mu_{\max}}\left[1 + \frac{(1+r)K_s}{(S_{in} - S^*)}\ln\left(\frac{S_{in} + rS^*}{S^*(1+r)}\right)\right] = \theta_0, \tag{22}$$

Since $r = r_0$ is constant, Equation (22) defines implicitly $S^* = S^*(S_{in})$, then Equation (16) gives $w = w(S_{in})$, Equation (14) gives $X_r = X_r(S_{in})$ and Equation (15) gives $X^* = X^*(S_{in})$.

## IV. ASP WITH DZC SETTLER MODEL

### A. Steady-state solutions

The mass balances of the system considering the DZC settler model have to include Equation (12) in order to get a sludge blanket in the thickening zone. The steady-state equations are then expressed as

$$S^* = S_{in} - \frac{w}{Y}U_{z_{sb}}(q), \tag{23}$$

$$X^* = \frac{r+w}{1+r}U_{z_{sb}}(q), \tag{24}$$

$$X_r = U_{z_{sb}}(q), \tag{25}$$

$$V_R = f(S^*, r, w). \tag{26}$$

Straightforward calculations give that the expression for the sludge age is the same as for the ideal settler model (cf. Equation (11) and [4])).

### B. Substrate input-output relation for constant sludge age

As in the case of an ASP with ideal settler, we are interested in the solution of the process for a constant sludge age for different values of $S_{in}$. By imposing $\theta(r, S_{in}, S^*) = \theta_0$ we get a one-parameter solution. Note that we cannot impose another equation (e.g. $r = r_0$) as we did for the ideal settler model, since we have Equation (12) controlling the sludge blanket.

Hence, Equations (22), (23) and (26) are solved for $S^* = S^*(S_{in})$, $r = r(S_{in})$ and $w = w(S_{in})$. Then, Equation (24) gives $X^* = X^*(S_{in})$ and Equation (25) gives $X_r = X_r(S_{in})$.

## V. AN APPROXIMATION FOR $S^*$ GIVEN $\theta_0$

Note that $S^*$ is given implicitly in Equation (22), and will depend on $S_{in}$, $r$ and $\theta_0$. This equation can, however, be solved explicitly for $S^*$ if we make an assumption.

**Theorem 2.** *Given an ASP described by Equations* (14)–(16) *(for an ideal settler) or by* (23)–(26) *(for a DZC settler), and where $\theta$ is given by* (11). *Assume that $\theta$ is fixed to $\theta_0$, i.e. Equation* (22) *is imposed. If $S_{in} \gg S^*$, then the following simple expression for $S^*$ holds:*

$$S^* = \frac{S_{in}}{(1+r)\exp(\alpha S_{in}) - r}, \tag{27}$$

*where*

$$\alpha = \frac{\theta_0\mu_{\max} - 1}{K_s(1+r)}.$$

*Proof.* Assuming $S_{in} \gg S^*$, Equation (22) can be written as

$$\frac{1}{\mu_{\max}}\left[1 + \frac{(1+r)K_s}{S_{in}}\ln\left(\frac{S_{in} + rS^*}{S^*(1+r)}\right)\right] = \theta_0,$$

solving for $S^*$ gives

$$S^* = \frac{S_{in}}{(1+r)\exp(\alpha S_{in}) - r},$$

where $\alpha = (\theta_0\mu_{\max} - 1)/(K_s(1+r))$. $\qquad\square$

## VI. NUMERICAL EXAMPLE

We assume that the ASP has the following constants and parameters: $V_R = 3000\ \text{m}^3$, $Q = 1000\ \text{m}^3/\text{h}$, $\mu_{\max} = 0.17\ \text{h}^{-1}$, $K_s = 0.05\ \text{kg/m}^3$, $Y = 0.7$. For the DZC settler model we let: $B = 3\ \text{m}$, $z_{sb} = 1\ \text{m}$, $X_1^\infty = 6.52\ \text{kg/m}^3$, $\hat{q}_1 = 0.32\ \text{m/h}$, $\check{q}_1 = 0.45\ \text{m/h}$. The latter constants were obtained with standard parameters for the hindered settling and compression functions and the procedure in [9].

Numerical solutions of the model equations will now be compared with the approximate solutions given by Theorems 1 and 2. The numerical solutions are obtained with `fsolve`, a function in Matlab which solves systems of non-linear equations.
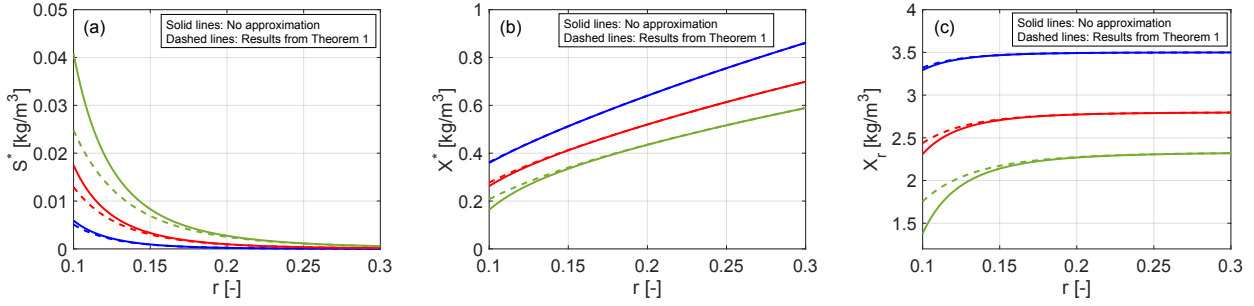
Fig. 3: PFR with ideal settler model. Comparison between the numerical (no approximation) and the approximated solutions given by Theorem 1 as functions of $r$. The results are shown for two values of the wastage: $w = 0.02$ (in blue), $w = 0.025$ (in red), $w = 0.03$ (in green). The influent substrate concentration is fixed to $S_{in} = 0.1 \, kg/m^3$.

### A. Theorem 1

This case deals with an ASP with an ideal settler model. Figure 3 shows the numerical (without approximation, i.e. without the assumption $S_{in} \gg S^*$) and approximated solutions given by Theorem 1 for $S^*, X^*$ and $X_r$. That is, we compare the results from Equations (14)–(16) with results from Equations (17)–(19). The influent substrate concentration is set to $S_{in} = 0.1 \, kg/m^3$. The results are shown for an interval of values of $r$ and for some values of the wastage ratio $w$.

Note that in plot (a), for higher values of $S^*$, the difference between the values given by Theorem 1 and the values from the solution with no approximation becomes larger. The same effect can be seen in plot (b) for $X^*$ and in plot (c) for $X_r$. $S^*$ starts to decrease for higher values in $r$. Then, the difference between values with and without approximation starts to decrease.

### B. Theorem 2

For the ideal settler model, Figure 4 shows numerical and approximated solutions given by Theorem 2. Equation (22) is solved for $S^* = S^*(S_{in})$ at an interval of values for $S_{in}$. The recirculation is set to $r = r_0 = 1$ (cf. Equation (21)), and we set $\theta_0 = 16 \, h$. Note that, for higher values of $S_{in}$, the values for $S^*$ given by Theorem 2 are closer to those given by the solution of the process with no approximation.

For the DZC settler model, Equations (22), (23) and (26) are solved for $S^* = S^*(S_{in}), r = r(S_{in})$ and $w = w(S_{in})$ at an interval of values for $S_{in}$. Figure 5 shows the numerical and approximated solutions for $S^* = S^*(S_{in})$. We set $\theta_0 = 6.5 \, h$ and show some results for some values of the settler area $A_S$.

Note that, for a given $S_{in}$ and when a higher $A_S$ is used, the solution with no approximation gives a higher recycle concentration $X_r$ (see Equations (12) and (13)). This means that we have a more thickened sludge, which gives a lower $S^*$ (see Equation (23)). Therefore, $S^*$ becomes much lower compared to $S_{in}$ as $A_S$ increases. Hence, the values from Theorem 2 are much closer to the solution of the model equations without approximation.
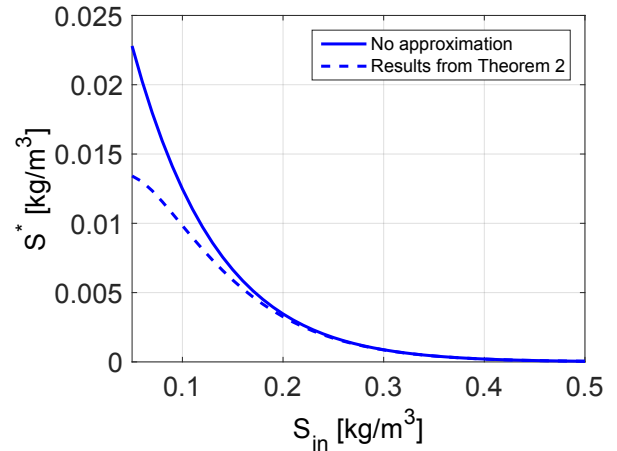


Fig. 4: PFR with ideal settler model. Comparison between the numerical and the approximated solutions given by Theorem 2 as functions of $S_{in}$. The recirculation is fixed to $r_0 = 1$ and the sludge age is kept to $\theta_0 = 16 \, h$.
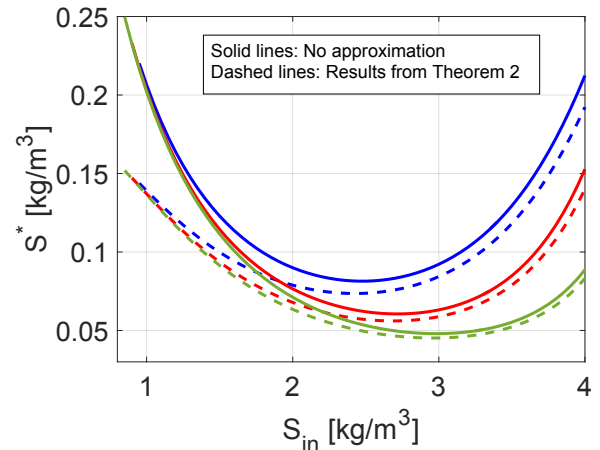


Fig. 5: PFR with DZC settler model. Comparison between the numerical (no approximation) and the approximated solutions given by Theorem 2 as functions of $S_{in}$ for some values of the settler area $A_S$ [m²]: 500 m² (in blue), 1500 m² (in red), 3000 m² (in green). For every curve, the sludge age is kept to $\theta_0 = 6.5 \, h$.

### VII. CONCLUSIONS

An ASP formed by a PFR and a settler has been studied in steady-state operation. It is shown that explicit, but

approximate, solutions can be obtained for the case of an ideal settler under the assumption that the influent substrate concentration is much greater than the effluent one. With this assumption it is also possible to obtain an explicit expression for the effluent concentration as a function of the influent one under the operating condition that the sludge age should be maintained at a specific value. Numerical examples show the performances of the simpler explicit expressions under two different models for the settler and hence when the simpler formulas can be used. Further research might be focused on considering the decayed particulate biomass as an additional constituent in the process.

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